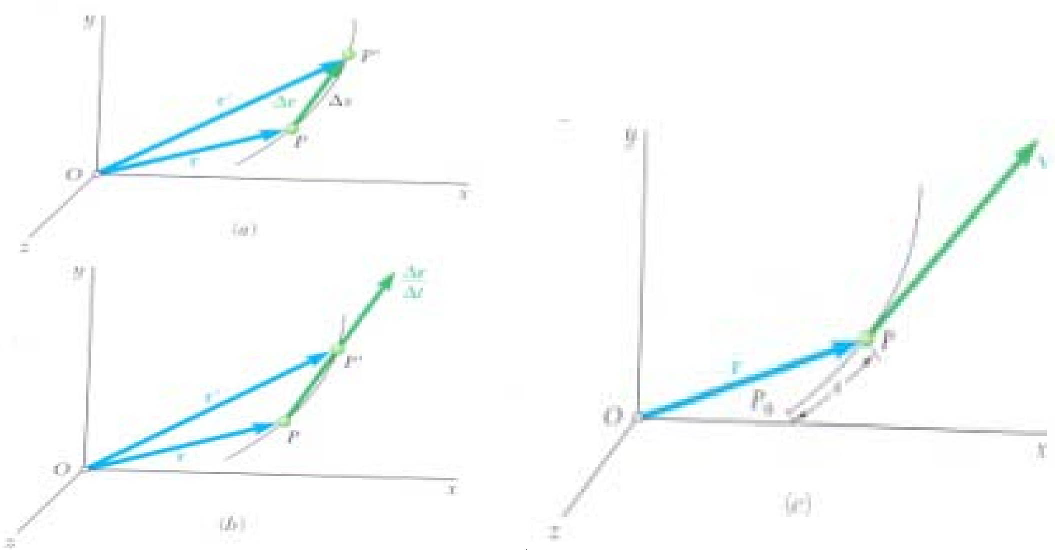
WEEK 2

GEC 241- APPLIED MECHANICS II- DYNAMICS
CURVILINEAR MOTION OF PARTICLES

POSITION VECTOR, VELOCITY AND ACCELERATION

When a particle moves along a curve other than a straight line, the particle is said to be in curvilinear motion. Hence to define the position P occupied by the particle at a given time t, a fixed reference system such as x,y,z axes are drawn as shown in the figures below



Vector **r** is characterized by its magnitude, r, and its direction with respect to the reference axes; it completely defines the position of the particle with respect to those axes. Hence the vector **r** is referred to as the position vector of the particle at time t.

When the position of P moves to P' at a later time $t + \Delta t$, vector r^1 is defined. The vector $\Delta \boldsymbol{r}$ joining P and P' represents the change in the position vector during the time interval Δt.

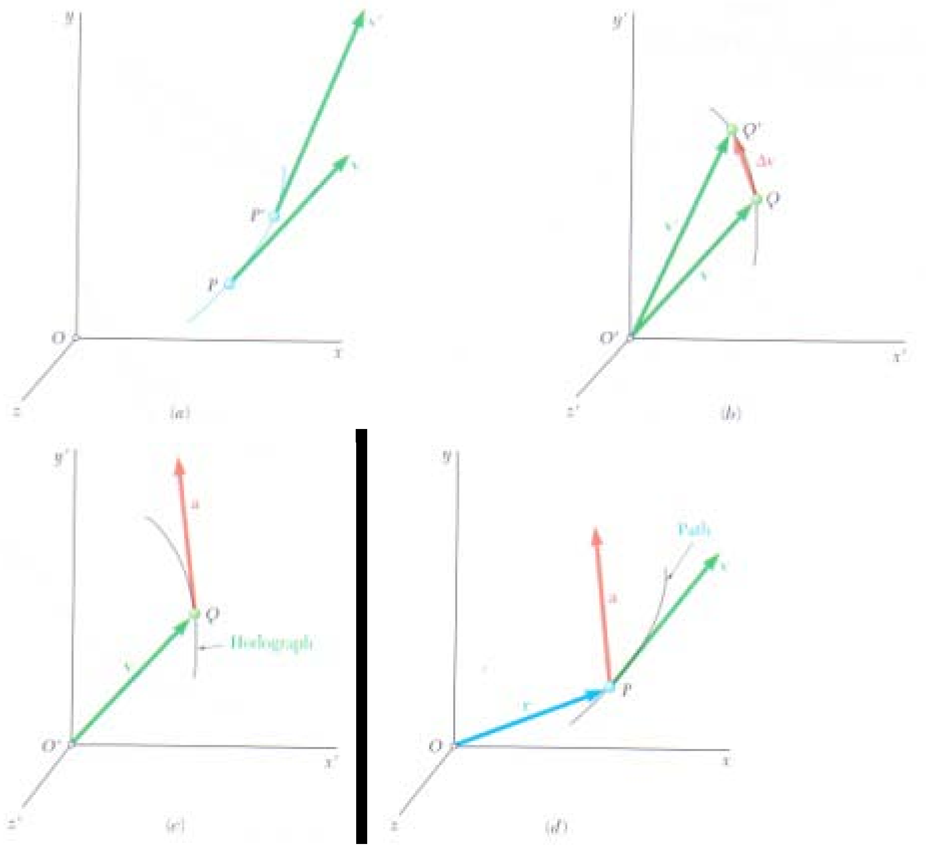
The average velocity of the particle over the time interval Δt is defined as the quotient of $\Delta r \; and \; \Delta t$.

The instantaneous velocity of the particle at time t is obtained by choosing shorter and shorter time intervals Δt and correspondingly shorter and shorter vector increments Δr. The instantaneous velocity is thus represented by the vector

$$v = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} \qquad (1)$$

Note that as Δt and Δr becomes shorter, the points P and P' get closer.

which can be written as $$v = \frac{dr}{dt} \qquad (2)$$



The average acceleration of a particle over the time interval Δt is defined as the quotient of Δv and Δt.

The instantaneous acceleration of a particle at time t is obtained by choosing smaller and smaller values for Δt and Δv. The instantaneous acceleration is thus represented by the vector

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} \qquad (3)$$

which can be written as

$$a = \frac{dv}{dt} \qquad (4)$$

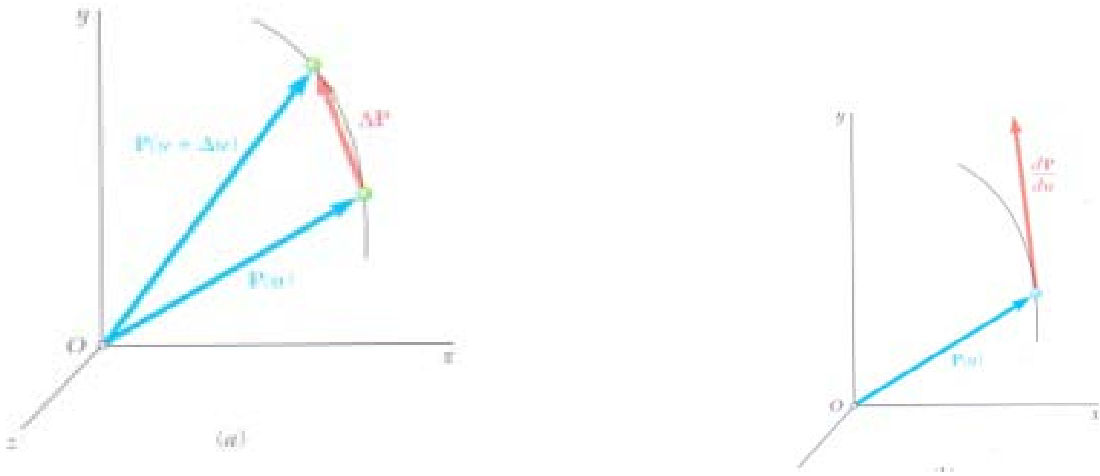
DERIVATION OF VECTOR FUNCTIONS

Let P(u) be the rectangular components of a derivative of a vector function in x,y,z axes.

$$P = P_x \boldsymbol{i} + P_y \boldsymbol{j} + P_z \boldsymbol{k} \qquad (\mathbf{5})$$

where P_x, P_y, P_z are the rectangular scalar components of the vector P and I,J.K the unit vectors corresponding respectively to the x, y and z axes.

The derivative of P is equal to the sum of the derivatives of the terms in the right-hand member.



$$\frac{dP}{du} = \frac{dP_x}{du}\boldsymbol{i} + \frac{dP_y}{du}\boldsymbol{j} + \frac{dP_z}{du}\boldsymbol{k} \qquad (\mathbf{6})$$

Rate of change of a vector

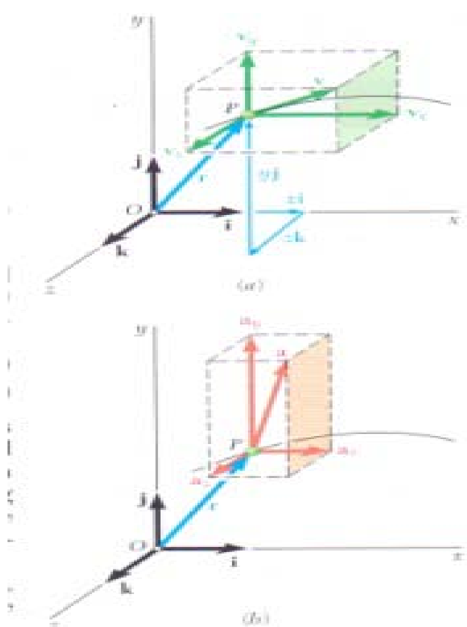
This occurs when the vector p is a function of the time t, its derivative dP/dt represents the rate of change of P with respect to the frame Oxyz. i.e.

$$\frac{dP}{dt} = \frac{dP_x}{dt}\boldsymbol{i} + \frac{dP_y}{dt}\boldsymbol{j} + \frac{dP_z}{dt}\boldsymbol{k} \qquad (\mathbf{7a})$$

Or $$\dot{P} = \dot{P}x\boldsymbol{i} + \dot{P}y\boldsymbol{j} + \dot{P}z\boldsymbol{k} \qquad (\mathbf{7b})$$

RECTANGULAR COMPONENTS OF VELOCITY AND ACCELERATION

It will be convenient to resolve the velocity v and the acceleration a of the particle into rectangular components when the position of a particle P has been defined at any instant by its rectangular coordinates x,y and z.



Resolving the position vector r of the particle into rectangular components, we have

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \qquad (8)$$

Where x,y,z are functions of t. Differentiating twice, we obtain

$$v = \frac{d\boldsymbol{r}}{dt} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k} \qquad (9)$$

$$a = \frac{dv}{dt} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{z}\boldsymbol{k} \qquad (10)$$

Where $\dot{x}$, $\dot{y}, \dot{z}$ and $\ddot{x}, \ddot{y}$, $\ddot{z}$ represent, respectively, the first and second derivatives of x,y and z with respect to t. i.e.

$$v_x = \dot{x} \qquad v_y = \dot{y} \qquad v_z = \dot{z}$$

$$a_x = \ddot{x} \qquad a_y = \ddot{y} \qquad a_z = \ddot{z}$$

The use of the rectangular components to describe the position, the velocity, and the acceleration of a particle is particularly effective when the component a_x of the acceleration depends only upon t, x and /or v_x, and when similarly a_y depends only upon t, y, and/or v_y $and\ a_z\ upon\ t, z$ and/or v_z.

In other words, the motion of the particle in the x- direction, its motion in the y direction and its mtion in its z direction can be considered separately.

For instance, in the motion of a projectile, the components of the acceleration are

$$a_x = \ddot{x} = 0 \qquad a_y = \ddot{y} - g \qquad a_z = \ddot{z} = 0$$

If the air resistance is neglected. Denoting by $x_0, y_0\ and\ z_0$ the coordinate of gun and by $(v_x)_0, (v_y)_0\ and\ (v_z)_0$the components of the initial velocity v_0 of the projectile (a bullet), we integrate twice in t and obtain

$$v_x = \dot{x} = (v_x)_0 \qquad v_y = \dot{y} = (v_y)_0 - gt \qquad v_z = \dot{z} = (v_z)_0$$

$$x = x_o + (v_x)_0 t \qquad y = y_o + (v_y)_0 t - \frac{1}{2}gt^2 \qquad z = z_o + (v_z)_0 t$$

If the projectile is fired in the xy plane from the origin O, we have $x_0 = y_0 = z_0 = 0\ and\ (v_z)_0 = 0$. The equations of motion reduce to

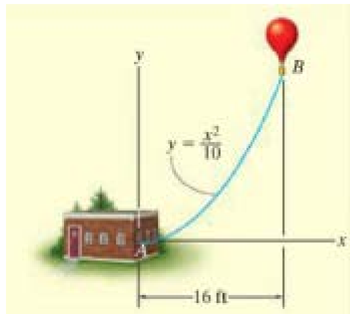
$$v_x = (v_x)_0 \qquad v_y = (v_y)_0 - gt \qquad v_z = 0$$

$$x = (v_x)_0 t \qquad y = (v_y)_0 t - \frac{1}{2}gt^2 \qquad z = 0$$

These equations show that the projectile remains in the xy plane.

Example 1

At any instant the horizontal position of the weather ballon in the figure below is defined by $x = (8t)ft$, where t is in seconds. If the equation of the path is $y = {x^2}/{10}$, determine the magnitude and direction of the velocity and the acceleration when t = 2 s.



SOLUTION

Velocity. The velocity of the component in the x-direction is $v_x = \dot{x} = \frac{d}{dt}(8t) = 8\,ft/s \rightarrow$

To find the relationship between the velocity components, chain rule of calculus will be used. When t=2 s, x= 8(2)= 16 ft and so

$$v_y = \dot{y} = \frac{d}{dt}\left({x^2}/{10}\right) = 2x\dot{x}/10 = \frac{2(16)(8)}{10} = 25.6\,ft/s \uparrow$$

When t = 2 s, the magnitude of velocity is therefore

$$v = \sqrt{8^2 + 25.6^2} = 26.8\,ft/s$$

The direction is tangent to the path where $\theta_v = tan^{-1}\frac{v_y}{v_x} = tan^{-1}\frac{25.6}{8} = 72.6^0$

Acceleration: The relationship between the acceleration components is determined using the chain rule

$$a_x = \dot{v_x} = \frac{d}{dt}(8) = 0$$

$$a_y = \dot{v_y} = \frac{d}{dt}\left(\frac{2x\dot{x}}{10}\right) = 2(\dot{x})\dot{x}/10 + 2x\ddot{x}/10$$

$$= {2(8)^2}/{10} + {2(16)(0)}/{10} = 12.8\,ft/s^2 \uparrow$$

Thus, $a = \sqrt{0^2 + 12.8^2} = 12.8\,ft/s^2$

The direction of **a** is $\theta_a = tan^{-1}\frac{a_y}{a_x} = tan^{-1}\frac{12.8}{0} = 90^0$

Example 2

For a short time, the path of an airplane is described by $y = (0.001x^2)\ m$. If the plane is rising with a constant upward velocity of 10m/s, determine the magnitudes of the velocity and acceleration of the plane when it reaches an altitude of y=100 m.

SOLUTION

When y = 100m, then 100 = $0.001x^2$ or x = 316.2 m., $v_y = 10\,m/s$

$$y = v_y t; \qquad 100\ m = (10\,m/s)t \qquad t = 10\ s$$

Velocity. Using the chain rule to find the relationship between the velocity components, we have

$$y = (0.001x^2)$$

$$v_y = \dot{y} = \frac{d}{dt}(0.001x^2) = (0.002x)\dot{x} = 0.002xv_x \qquad (1)$$

$$thus, 10\,m/s = 0.002(316.2m)(v_x)$$

$$v_x = 15.81\,m/s$$

The magnitude of the velocity is $v = \sqrt{15.81^2 + 10^2} = 18.7\,m/s$

Acceleration. Using the chain rule, the time derivative of the equation (1) above gives the relation between the acceleration components

$$a_y = \dot{v}_y = (0.002\dot{x})\dot{x} + 0.002x(\ddot{x}) = 0.002(v_x^2 + xa_x)$$

$$when\ x = 316.2m, v_x = 15.81\,m/s\,, \dot{v}_y = a_y = 0$$

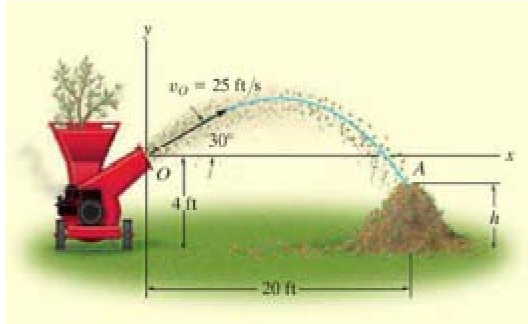
$$0 = 0.002[15.81^2 + 316.2(a_x)$$

$$a_x = -0.791\,m/s^2$$

The magnitude of the acceleration is $a = \sqrt{(-0.791)^2 + 0^2} = 0.791\,m/s^2$

Example 3

The chipping machine is designed to eject wood chips at $v_0 = 25\,ft/s$ as shown in the figure below. If the tube is oriented at 30^0 from the horizontal, determine how high,h, the chips strike the pile if at this instant they land on the pole 20 ft from the tube.



SOLUTION

Analyzing the motion between points O and A, the three unknowns are the height h, time of flight t_{OA}, and vertical component of velocity $(v_A)_y$.

Note that $(v_A)_x = (v_o)_x$

The initial velocity of a chip has components of

$$(v_o)_x = (25cos30^0)\, ft/s = 21.65\, ft/s \rightarrow$$

$$(v_o)_y = (25sin30^0)\, ft/s = 12.5\, ft/s \uparrow$$

Also $(v_A)_x = (v_o)_x = 21.65\, ft/s \ and\ a_y = -32.2\, ft/s^2\ (same\ as\ 9.81\, m/s^2)$

Horizontal motion.

$$x_A = x_o + (v_o)_x t_{OA}$$

$$20 = 0 + 21.65 t_{OA}$$

$$t_{OA} = 0.9238\ s$$

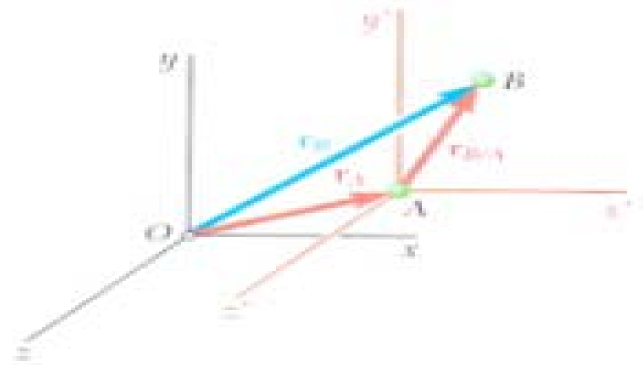
Vertical motion.

$$y_A = y_o + (v_o)_y t_{OA} + \frac{1}{2} a_c t_{OA}^2$$

$$h - 4 = 0 + (12.5)(0.9238) + \frac{1}{2}(-32.2)(0.9238)^2$$

$$h = 1.81 ft$$

RELATIVE MOTION IN A PLANE



Consider two particles A and B moving in space, the vectors $r_A andr_B$ define their position at any given instant with respect to the fixed frame of reference Oxyz. Consider now a system of axes $x^{'}, y^{'}, z^{'}$ centered at A and parallel to the x, y, z axes. While the origin of these axes moves, their orientation remains the same, the frame of reference A$x^{'}y^{'}z^{'}$ is in translation with respect to Oxyz. The vector $r_{B/A}$ joining A and B defines the position of B relative to the moving frame A$x^{'}y^{'}z^{'}$ or the position of B relative to A.

The position vector r_B of particle B is the sum of the position vector r_A of particle A and of the position vector $r_{B/A}$ of B relative to A.

$$r_B = r_A + r_{B/A} \qquad (11a)$$

Differentiate with respect to t, we have

$$v_B = v_A + v_{B/A} \qquad (11b)$$

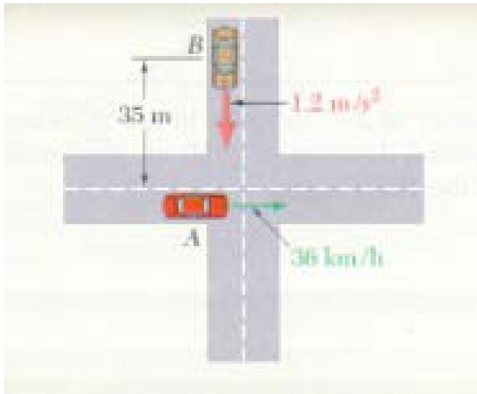
And differentiating the velocity, we have

$$a_B = a_A + a_{B/A} \qquad (11c)$$

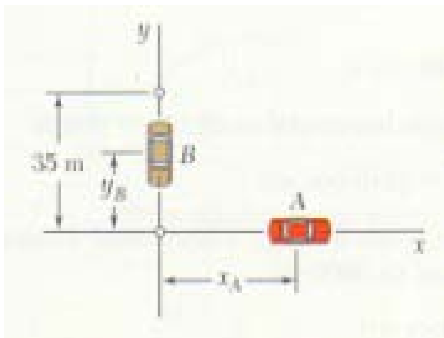
The motion of B with respect to the fixed frame Oxyz is referred to as the absolute motion of B. The equations derived show that the absolute motion of B can be obtained by combining the motion of A and the relative motion of B with respect to the frame attached to A.

Example 4

Automobile A is travelling east at a constant speed of 36km/h. As automobile B crosses the intersection shown, the automobile B starts from rest 35m north of the intersection and moves south with a constant acceleration of 1.2m/s^2. Determine the position, velocity and acceleration of B relative to A 5s after A crosses the intersection.



SOLUTION



Motion of Automobile A

The first speed $v_A = \frac{36 \times 1000}{3600} = 10\text{m/s}$

Note that the motion of A is uniform for any time t

$v_A = +10\text{mls}, a_A = 0$

$$x_A = (x_A)_0 + v_A t = 0 + 10t$$

For t = 5 s

$v_A = +10\overrightarrow{\text{mls}}, a_A = 0, x_A = 5\overrightarrow{0m}$

Motion of Automobile B- The motion of B is uniformly accelerated

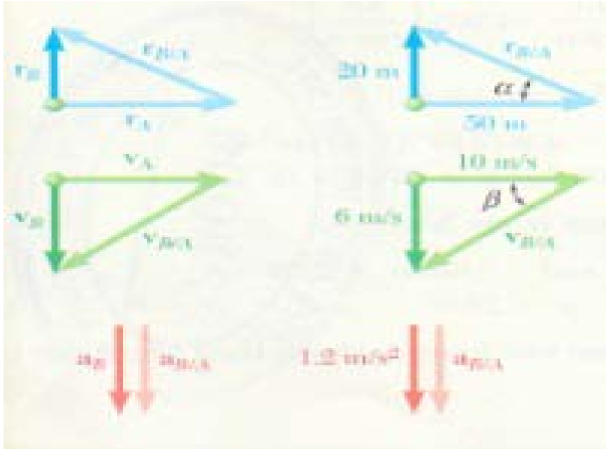
$a_B = -1.2\text{m/s},$

$$v_B = (x_B)_0 + at = 0 - 1.2t$$

$$y_B = (y_B)_0 + (v_B)_0 t + \frac{1}{2} a_B t^2 \;= 35 + 0 - \frac{1}{2}(1.2)t^2$$

For t $\downarrow$ = 5 s, $a_B = -1.2\text{m/s},$ $v_B = -1.2(5) = -6\text{m/s}\downarrow$

$y_B = 35 - \frac{1}{2}(1.2)(5)^2 = 20\ m \uparrow$



Motion of B relative to A

$$r_B = r_A + r_{B/A}$$

$$r_{B/A} = 53.9m$$

$\alpha = tan^{-1}\left(\frac{20}{50}\right) = 21.8^0$

$r_{B/A} = 53.9m$ 21.8^0

$$v_B = v_A + v_{B/A}$$

$v_{B/A} = 11.66m/$s and the direction $\beta = 31^0$

$v_{B/A}$ =11.66m/s 31.0^0

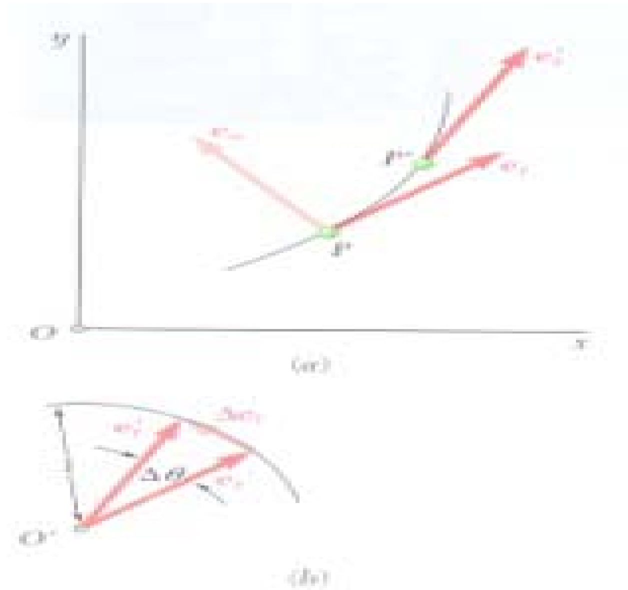
$a_{B/A}$= 1.2 m/s^2 $\downarrow$

TANGENTIAL AND NORMAL COMPONENTS

Note that the velocity of a particle is a vector tangent to the path of the particle but that in general, the acceleration is not tangent to the path. It is sometimes convenient to resolve the acceleration into components directed respectively, along the tangent and the normal to the path of the particle.

The tangential component of acceleration reflects a change in the speed of the particle while its normal component reflects a change in the direction of motion of the particle. The acceleration of particle will be zero only if both its components are zero. Thus the acceleration of a particle moving with constant speed along a curve will not be zero unless the particle happens to pass through a point of inflection of the curve (where the radius of curvature is infinite) or unless the curve is a straight line. The fact that the normal component of the acceleration depends upon the radius of curvature of the path followed by the particle is taken into account in the design of structures or mechanisms as widely different as airplane wings, rail road tracks, and cams.

Consider the diagrams below



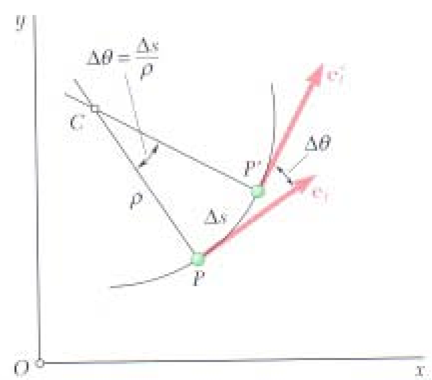
Let e_t be the unit vector corresponding to P and e'_t corresponding to P'. Drawing both vectors from the same origin O', we have $\Delta e_t = e'_t - e_t$.

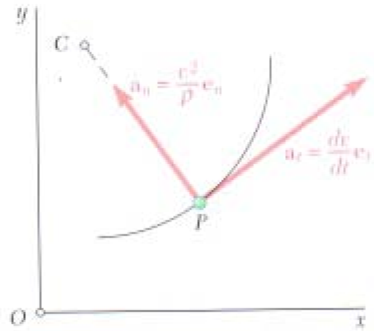
Considering the vector $\frac{\Delta e_t}{\Delta \theta}$, it was noted that as $\Delta\theta$ approaches zero, this vector becomes tangent to the unit circle i.e. perpendicular to e_t and that its magnitudes approaches

$$\lim_{\Delta\theta \to 0} \frac{2sin\left(\frac{\Delta\theta}{2}\right)}{\Delta\theta} = \lim_{\Delta\theta \to 0} \frac{2sin\left(\frac{\Delta\theta}{2}\right)}{\Delta\theta/2} = 1$$

Thus the vector obtained in the limit is a unit vector along the normal to the path of the particle, in the direction toward which e_t turns. Denoting this vector by e_n, we write

$$e_n = \lim_{\Delta\theta \to 0} \frac{\Delta e_t}{\Delta\theta} = \frac{de_t}{d\theta} \qquad (12)$$





Since the velocity v of the particle is tangent to the path it can be expressed as the product of the scalar v and the unit vector e_t.

$$V = ve_t \qquad (13)$$

To obtain the acceleration of the particle, differentiate with respect to t

$$a = \frac{dV}{dt} = \frac{dv}{dt}e_t + v\frac{de_t}{dt} \qquad (14)$$

But $\frac{de_t}{dt} = \frac{de_t}{d\theta}.\frac{d\theta}{ds}.\frac{ds}{dt}$

Recall that $\frac{ds}{dt} = v \ and \ \frac{de_t}{d\theta} = e_n$

Also from elementary calculus, $\frac{d\theta}{ds} = \frac{1}{p}$ where p is the radius of curvature of the path P.

Therefore, $\frac{de_t}{dt} = \frac{v}{p}e_n$ (15)

Substituting (15) into (14), we have

$$a = \frac{dv}{dt}e_t + \frac{v^2}{p}e_n \qquad (16)$$

The scalar components of the acceleration are $a_t = \frac{dv}{dt} \ and \ a_n = \frac{v^2}{p}$

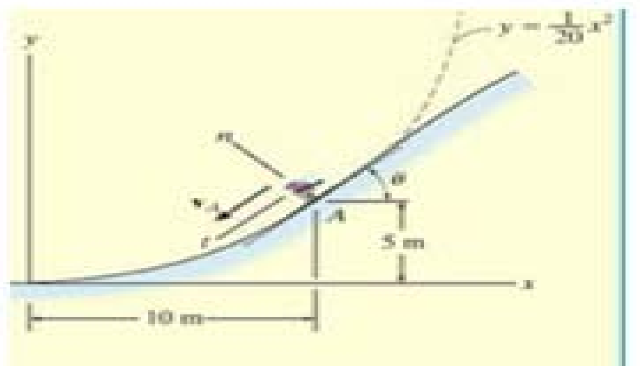
The relations obtained express that the tangential component of the acceleration is equal to the rate of change of the speed of the particle, while the normal component is equal to the square of the speed divided by the radius of curvature of the path at P.

Note that if the path is expressed as $y = f(x)$, the radius of curvature $p$at any point on the path is determined from the equation

$$p = \frac{[1+(dy/dx)^2]^{3/2}}{[d^2y/dx^2]}$$

Example 5

When the skier reaches point A along he parabolic path in the figure below, he has a speed of 6 m/s which is increasing at 2m/s^2. Determine the direction of his velocity and the direction and magnitude odf his acceleration at this instant. Neglect the size of the skier in the calculation.



SOLUTION

Velocity. The velocity is always directed tangent to the path. Since $y = \frac{1}{20}x^2, dy/dx = \frac{1}{10}x, then\ at\ x = 10\ m, dy/dx = 1$

Hence at A, **v** makes an angle of $\theta = tan^{-1}1 = 45^0$ with the x axis

$$v_A = 6\,m/s\ \ and\ the\ direction\ is\ 45^0$$

The acceleration is determined from $a = \frac{dv}{dt}e_t + \frac{v^2}{p}e_n$.

However, we need to determine the radius of curvature of the path at A using the formula

$$p = \frac{[1+(dy/dx)^2]^{3/2}}{[d^2y/dx^2]} = \frac{[1+(\frac{1}{10}x)^2]^{3/2}}{\frac{1}{10}} \text{ at x = 10 m}$$

= 28.28 m

Acceleration becomes

$$a_A = \frac{dv}{dt}e_t + \frac{v^2}{p}e_n$$

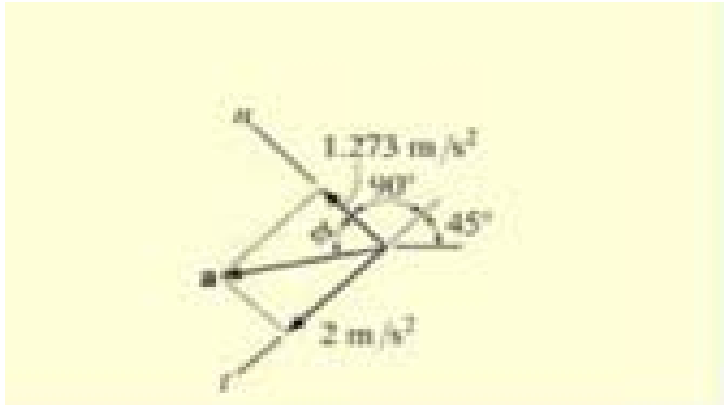
$$= 2e_t + \frac{6^2}{28.28}e_n$$

$$a = 2e_t + 1.273e_n \quad m/s^2$$

The magnitude of acceleration is

$$a = \sqrt{2^2 + 1.273^2} = 2.37\,m/s^2$$

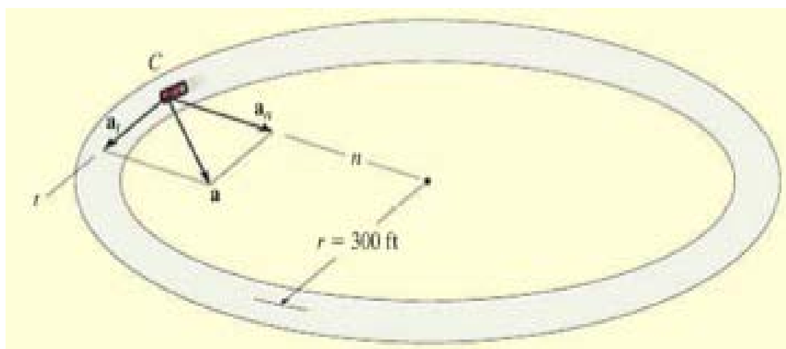
$$\emptyset = tan^{-1}\frac{2}{1.273} = 57.5^0$$



Thus the direction of the acceleration is $45^0+90^0+57.5^0-180^0 = 12.5^0$

Example 6

A race car C travels around the horizontal circular track that has a radius of 300 ft. If the car increases its speed at a constant rate of 7 ft/s^2, starting from rest, determine the time needed for it to reach an acceleration of 8 ft/s^2. What is its speed at this instant?



SOLUTION

Note that the origin of the n and t axes is coincident with the car at the instant considered. The t axis is in the direction of the motion, and the positive n axis is directed toward the centre of the circle. His coordinate system is selected since the path is known.

Acceleration.

The magnitude of the acceleration can be related to its components using $a = \sqrt{a_t^2 + a_n^2}$. Here $a_t = 7\,ft/s^2$. Since $a_n = \frac{v^2}{p}$, the velocity as a function of time must be determined first.

$$v = v_o + (a_t)_c t$$

$$v = 0 + 7t = 7t$$

Thus $a_n = \frac{v^2}{p} = \frac{(7t)^2}{300} = 0.163t^2\ ft/s^2$

The time needed for the acceleration to reach 8 ft/s^2 is therefore

$$a = \sqrt{a_t^2 + a_n^2}$$

$$8 = \sqrt{7^2 + (0.163t^2)^2}$$

t = 4.87 s

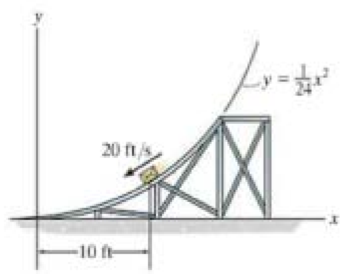
Velocity. The speed at time t = 4.87 s is

$$v = 7t = 7(4.87) = 34.1\ ft/s$$

NOTE: Remember the velocity will always be tangent to the path whereas the acceleration will be directed with the curvature of the path.

PRACTICE QUESTIONS

1. A particle is travelling along a straight path described as $= 0.75x$. If its position along the x axis is $x = (8t)m$, where t is in seconds, determine its speed when t = 2 s.
2. A particle is constrained to travel along the path. If $x = (4t^2)m$, where t is in seconds, determine the magnitude of the particle's velocity and acceleration when t= 0.5 s.
3. A particle traveling along a parabolic path $y = 0.25x^2$. If $x = (2t^2)m$, where t is in seconds, determine the magnitude of the particle's velocity and acceleration when t = 2s.
4. When x = 10 ft, the crate has a speed of 20 ft/s which is increasing at 6 ft/s^2. Determine the direction of the crate's velocity and the magnitude of the crate's acceleration at this instant.



5. A boat is sailing along a circular path of $v = (0.0625t^2)\ m/s$, where t is in seconds. Determine the magnitude of its acceleration when t= 10 s.